Group theory approach to the Dirac equation with a Coulomb plus scalar potential in $D+1$ dimensions[a]

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We generalize the Dirac equation to $D+1$ space–time. The conserved angular momentum operators and their quantum numbers are discussed. The eigenfunctions of the total angular momentums are calculated for both odd D and even D cases. The exact solutions of the $D+1$-dimensional radial equations of the Dirac equation with a Coulomb plus scalar potential are analytically presented by studying the Tricomi equations obtained from a pair of coupled first-order ones. The eigenvalues are also discussed in some detail. © *2003 American Institute of Physics.*
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I. INTRODUCTION

The exact solutions of the nonrelativistic and relativistic equations with the Coulomb potential play an important role in quantum mechanics.[1–3] For example, the study of the exact solutions of the Schrödinger equation for a hydrogen atom is an important advance at the beginning of establishment of quantum mechanics. Recently the study of the Dirac equation with the Coulomb plus scalar potential has been investigated. For instance, the bound states of this case have been studied in $3+1$ dimensions.[4,5] Moreover the corresponding S-matrix in the quantum scattering theory has also been carried out by Vaidya and Souza in $3+1$ dimensions.[6] With the interest of the lower-dimensional field theory and condensed matter physics, the lower-dimensional case seems physically relevant since the results obtained in this case exhibit some new features. Therefore, the bound states of the $(2+1)$-dimensional Dirac equation with the Coulomb plus scalar potential have been investigated in our previous work.[7] Similarly with the interest of the higher-dimensional field theory, it is worth studying the exact solutions of this quantum system in $D+1$ dimensional space–time, which is the main purpose of this work.

This article is organized as follows. Section II is devoted to the generalization of the Dirac equation to $D+1$ space–time. In Sec. III, the conserved angular momentum operators and their quantum numbers are discussed. The eigenfunctions of the total angular momentums are calculated for both odd D and even D cases from the view point of the group theory. The radial equations of this quantum system are obtained. In Sec. IV, the exact solutions of the radial equations, which are expressed by the confluent hypergeometric functions, are analytically presented. The energy levels and some special cases are also discussed in great detail. The concluding remarks are given in Sec. V.

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II. THE DIRAC EQUATION IN $D+1$ DIMENSIONS

In this section we review some properties of the Dirac equation in $D+1$ dimensional space–time. The Dirac equation in $D+1$ dimensions can be written as[8]

$$i\sum_{\mu=0}^{D} \gamma^{\mu}(\partial_{\mu}+ieA_{\mu})\Psi(\mathbf{x},t)=M\Psi(\mathbf{x},t), \tag{1}$$

where M is the mass of the particle, and $D+1$ matrices γ_{μ} satisfy the anticommutative relations:

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu}=2\eta^{\mu\nu}\mathbf{1}, \tag{2}$$

with

$$\eta^{\mu\nu}=\eta_{\mu\nu}=\begin{cases}\delta_{\mu\nu} & \text{when} \quad \mu=0,\\ -\delta_{\mu\nu} & \text{when} \quad \mu\neq 0.\end{cases} \tag{3}$$

Throughout this article, the natural units $\hbar=c=1$ are employed if not explicitly stated otherwise. Discuss the special case where only A_0 of A_{μ} is nonvanishing and spherically symmetric:

$$eA_0=V(r), \qquad A_a=0, \qquad \text{when} \quad a\neq 0. \tag{4}$$

The Hamiltonian $H(\mathbf{x})$ of the system is expressed as

$$\begin{gathered} i\partial_0\Psi(\mathbf{x},t)=H(\mathbf{x})\Psi(\mathbf{x},t), \qquad H(\mathbf{x})=\sum_{c=1}^{D}\gamma^0\gamma^c p_c+V(r)+\gamma^0 M, \\ p_c=-i\partial_c=-i\frac{\partial}{\partial x^c}, \quad c\in[1,D]. \end{gathered} \tag{5}$$

The orbital angular momentum operators L_{ab}, the spinor operators S_{ab}, and the total angular momentum operators J_{ab} are defined as follows:

$$\begin{gathered} L_{ab}=-L_{ba}=ix_a\partial_b-ix_b\partial_a, \quad S_{ab}=-S_{ba}=i\frac{\gamma_a\gamma_b}{2}, \\ J_{ab}=L_{ab}+S_{ab}, \quad 1\leqslant a<b\leqslant D, \\ J^2=\sum_{a<b=2}^{D}J_{ab}^2, \quad L^2=\sum_{a<b=2}^{D}L_{ab}^2, \quad S^2=\sum_{a<b=2}^{D}S_{ab}^2. \end{gathered} \tag{6}$$

The eigenvalue of J^2 (L^2 or S^2) is denoted by the Casimir $C_2(\mathbf{M})$, where $\mathbf{M}$ is the highest weight of the representation to which the total (orbital or spinor) wave function belongs. We will discuss the Casimir in the next section. It is easy to show by the standard method[8] that J_{ab} and κ are commutative with the Hamiltonian $H(\mathbf{x})$,

$$\kappa=\gamma^0\left(\sum_{a<b} i\gamma^a\gamma^b L_{ab}+\frac{D-1}{2}\right)=\gamma^0\left(J^2-L^2-S^2+\frac{D-1}{2}\right). \tag{7}$$

III. THE RADIAL EQUATIONS

Because of the spherically symmetric potential $V(r)$, the symmetry group of the system is SO(D) group. Erdelyi,[9] Louck[10] and Chatterjee[11] have introduced the hyperspherical coordinates in the real D-dimensional space:

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$$
\begin{aligned}
x^1 &= r \cos\theta_1 \sin\theta_2 \cdots \sin\theta_{D-1}, \\
x^2 &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{D-1}, \\
x^b &= r \cos\theta_{b-1} \sin\theta_k \cdots \sin\theta_{D-1}, \qquad b\in[3,D-1], \\
x^D &= r \cos\theta_{D-1}, \\
\sum_{a=1}^{D} (x^a)^2 &= r^2.
\end{aligned} \tag{8}
$$

The unit vector along $\mathbf{x}$ is usually denoted by $\hat{\mathbf{x}}=\mathbf{x}/r$. The volume element of the configuration space is

$$
\prod_{a=1}^{D} dx^a = r^{D-1} dr d\Omega, \quad d\Omega=\prod_{a=1}^{D-1} (\sin\theta_a)^{a-1} d\theta_a,
$$
$$
r\in[0,\infty], \qquad \theta_1\in[-\pi,\pi], \quad \theta_c\in[0,\pi], \qquad c\in[2,D-1]. \tag{9}
$$

We now sketch some necessary information of the SO(D) group. From the representation theory of Lie groups,[12–14] the Lie algebras of the groups SO($2N+1$) and SO($2N$) are B_N and D_N, respectively. Their Chevalley bases with the subscript μ, $1\leqslant\mu\leqslant N-1$, are the same:

$$
\begin{aligned}
H_\mu(J) &= J_{(2\mu-1)(2\mu)} - J_{(2\mu+1)(2\mu+2)}, \\
E_\mu(J) &= \tfrac{1}{2}(J_{(2\mu)(2\mu+1)} - J_{(2\mu-1)(2\mu+2)} - iJ_{(2\mu-1)(2\mu+1)} - iJ_{(2\mu)(2\mu+2)}), \\
F_\mu(J) &= \tfrac{1}{2}(J_{(2\mu)(2\mu+1)} - J_{(2\mu-1)(2\mu+2)} + iJ_{(2\mu-1)(2\mu+1)} + iJ_{(2\mu)(2\mu+2)}).
\end{aligned} \tag{10a}
$$

However, the bases with the subscript N are different:

$$
\begin{aligned}
H_N(J) &= 2J_{(2N-1)(2N)}, \\
E_N(J) &= -iJ_{(2N-1)(2N+1)} + J_{(2N)(2N+1)}, \\
F_N(J) &= iJ_{(2N-1)(2N+1)} + J_{(2N)(2N+1)}
\end{aligned} \tag{10b}
$$

for SO($2N+1$), and

$$
\begin{aligned}
H_N(J) &= J_{(2N-3)(2N-2)} + J_{(2N-1)(2N)}, \\
E_N(J) &= \tfrac{1}{2}(J_{(2N-2)(2N-1)} + J_{(2N-3)(2N)} + iJ_{(2N-2)(2N)} - iJ_{(2N-3)(2N-1)}), \\
F_N(J) &= \tfrac{1}{2}(J_{(2N-2)(2N-1)} + J_{(2N-3)(2N)} + iJ_{(2N-3)(2N-1)} - iJ_{(2N-2)(2N)}),
\end{aligned} \tag{10c}
$$

for SO($2N$). The operator J_{ab} may be replaced by L_{ab} or S_{ab} depending on the studied wave functions. $H_\mu(J)$ span the Cartan subalgebra, and their eigenvalues for an eigenstate $|\mathbf{m}\rangle$ in a given irreducible representation (IR) are the components of a weight vector $\mathbf{m}=(m_1,\ldots,m_n)$:

$$
H_\mu(J)|\mathbf{m}\rangle = m_\mu|\mathbf{m}\rangle, \qquad \mu\in[1,N]. \tag{11}
$$

If the eigenstates $|\mathbf{m}\rangle$ for a given weight $\mathbf{m}$ are degeneracy, this weight is called a multiple weight, otherwise a simple one. E_μ are called the raising operators and F_μ the lowering ones. For an IR there is a highest weight $\mathbf{M}$, which is a simple weight and is used to describe the IR. Generally, the

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irreducible representation is also called the highest weight representation and directly denoted by $\mathbf{M}$. The Casimir $C_2(\mathbf{M})$ is calculated by the formula [see (1.131) of Ref. 14]

$$C_2(\mathbf{M})=\mathbf{M}\cdot(\mathbf{M}+2\varrho)=\sum_{\mu,\nu=1}^{N} M_\mu d_\mu (A^{-1})_{\mu\nu}(M_\nu+2), \tag{12}$$

where ϱ is the half sum of the positive roots in the Lie algebra, A^{-1} is the inverse of the Carton matrix, and d_μ are the half square lengths of the simple roots.

The orbital wave functions in D-dimensional space are usually expressed by the spherical harmonics $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$,[10,11] which belong to the weight $\mathbf{m}$ of the highest weight representation $(l)\equiv(l,0,\ldots,0)$. For the highest weight state, $\mathbf{m}=(l)$, we have

$$Y^{(l)}_{(l)}(\hat{\mathbf{x}})=N_{D,l}r^{-l}(x^1+ix^2)^l, \tag{13a}$$

with the normalization factor

$$N_{D,l}=\begin{cases} 2^{-N-l}\sqrt{\dfrac{(2l+2N-1)!}{\pi^N l!(l+N-1)!}} & \text{when} \quad D=2N+1, \\ \sqrt{\dfrac{(l+N-1)!}{2\pi^N l!}} & \text{when} \quad D=2N. \end{cases} \tag{13b}$$

Its partners $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ is calculated from $Y^{(l)}_{(l)}(\hat{\mathbf{x}})$ by lowering operators $F_\mu(L)$. The Casimir for the spherical harmonic $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ can be calculated by Eq. (12):

$$L^2 Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})=C_2[(l)]Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}}), \tag{14}$$

with

$$C_2[(l)]=l(l+D-2).$$

It is known that the spinor wave functions as well as those for the total angular momentum are different for $D=2N+1$ and $D=2N$, as studied in Ref. 15. Nevertheless, for completeness and clearness, it is necessary to review how to calculate these wave functions with the help of the groups SO$(2N+1)$ and SO$(2N)$.

A. The SO$(2N+1)$ case

When $D=2N+1$ we can define

$$\gamma^0=\sigma_3\times\mathbf{1}, \qquad \gamma^a=(i\sigma_2)\times\alpha_a, \qquad a\in[1,2N+1], \tag{15}$$

with the Pauli matrix σ_a, the 2^N-dimensional unit matrix $\mathbf{1}$ and the $(2N+1)$ matrices α_a satisfying the following anticommutative relations:

$$\alpha_b\alpha_a+\alpha_a\alpha_b=2\delta_{ab}\mathbf{1}, \quad b,\ a=1,\ 2,\ldots,(2N+1). \tag{16}$$

The dimensions of α_a matrices are 2^N. Thus the spinor operator S_{ab} becomes a block matrix

$$S_{ab}=\mathbf{1}\times\overline{S}_{ab}, \quad \overline{S}_{ab}=-i\frac{\alpha_a\alpha_b}{2}. \tag{17}$$

The relation between S_{ab} and $\overline{S}_{ab}$ is very similar to that between the spinor operators for the Dirac spinors and for the Pauli spinors. The operator κ becomes

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$$\kappa = \sigma_3 \times \overline{\kappa}, \quad \overline{\kappa} = -i \sum_{a<b} \alpha_a \alpha_b L_{ab} + \frac{D-1}{2}. \tag{18}$$

The spinor $\chi(\mathbf{m})$ belongs to the spinor representation $(s) \equiv (0,\ldots,0,1)$. It is found from Eq. (12) that the Casimir for the representation (s) can be calculated as $C_2[(s)] = (2N^2+N)/4$.

On the other hand, it is well known that the product of $Y_{\mathbf{m}}^{(l)}(\hat{\mathbf{x}})$ and $\chi(\mathbf{m}')$ belongs to the direct product of two representation (l) and (s), which is a reducible representation:

$$(l) \times (s) \simeq (l,0,\ldots,0,1) \oplus (l-1,0,\ldots,0,1). \tag{19}$$

Generally speaking, there are two different ways to construct a wave function belonging to the representation $(j) \equiv (l,0,\ldots,0,1)$, namely, the combination of $Y_{\mathbf{m}}^{(l)}(\hat{\mathbf{x}})\chi(\mathbf{m}')$ and that of $Y_{\mathbf{m}}^{(l+1)}(\hat{\mathbf{x}})\chi(\mathbf{m}')$, which are different in eigenvalues of $\overline{\kappa}$. Considering the spherically symmetric system, we only calculate the highest weight state for the representation (j) from the Clebsch–Gordan coefficients

$$\phi_{|K|,(j)}(\hat{\mathbf{x}}) = Y_{(l)}^{(l)}(\hat{\mathbf{x}})\chi[(s)] = N_{D,l} r^{-l}(x^1 + ix^2)^l \chi[(s)], \tag{20}$$

with

$$|K| = C_2[(j)] - C_2[(l)] - C_2[(s)] + N = l + N,$$

and

$$\begin{aligned}
\phi_{-|K|,(j)}(\hat{\mathbf{x}}) &= \sum_{\mathbf{m}} Y_{\mathbf{m}}^{(l+1)}(\hat{\mathbf{x}})\chi[(j)-\mathbf{m}]\langle (l+1),\mathbf{m},(s),(j)-\mathbf{m}|(j),(j)\rangle \\
&= N_{D,l} r^{-l-1}(x^1+ix^2)^l \{x^{2N+1}\chi[(s)] + (x^{2N-1}+ix^{2N})\chi[(0,\ldots,0,1,\overline{1})] \\
&\quad + (x^{2N-3}+ix^{2N-2})\chi[(0,\ldots,0,1,\overline{1},1)] + \cdots + (x^3+ix^4)\chi[(1,\overline{1},0,\ldots,0,1)] \\
&\quad + (x^1+ix^2)\chi[(\overline{1},0,\ldots,0,1)]\},
\end{aligned} \tag{21}$$

with

$$-|K| = C_2[(j)] - C_2[(l+1)] - C_2[(s)] + N = -l - N.$$

The wave functions $\Psi_{K,(j)}(\mathbf{x})$ of the total angular momentum belonging to the IR (j) can be written as

$$\Psi_{K,(j)}(\mathbf{x},t) = r^{-N} e^{-iEt} \begin{pmatrix} F(r)\phi_{K,(j)}(\hat{\mathbf{x}}) \\ iG(r)\phi_{-K,(j)}(\hat{\mathbf{x}}) \end{pmatrix}, \tag{22a}$$

with the following properties:

$$\begin{gathered}
H_1(J)\Psi_{K,(j)}(\mathbf{x}) = l\Psi_{K,(j)}(\mathbf{x}), \\
H_N(J)\Psi_{K,(j)}(\mathbf{x}) = \Psi_{K,(j)}(\mathbf{x}), \\
H_\mu(J)\Psi_{K,(j)}(\mathbf{x}) = 0, \qquad \mu \in [2,N-1], \\
\kappa\Psi_{K,(j)}(\mathbf{x}) = K\Psi_{K,(j)}(\mathbf{x}), \quad K = \pm(l+N).
\end{gathered} \tag{22b}$$

Their partners can be calculated by the lowering operators F_μ.

The radial equation depends on the explicit forms of α_a matrices, which can be expressed by direct products of N Pauli matrices σ_a:[16]

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$$\alpha_{2m-1}=\overbrace{\mathbf{1}\times\ldots\times\mathbf{1}}^{m-1}\times\sigma_1\times\overbrace{\sigma_3\times\ldots\times\sigma_3}^{N-m},$$

$$\alpha_{2m-1}=\overbrace{\mathbf{1}\times\ldots\times\mathbf{1}}^{m-1}\times\sigma_2\times\overbrace{\sigma_3\times\ldots\times\sigma_3}^{N-m}, \tag{23}$$

$$\alpha_{2N+1}=\sigma_3\times\sigma_3\times\ldots\times\sigma_3.$$

From the explicit forms of α_a, one can obtain

$$(\vec{\alpha}\cdot\hat{\mathbf{x}})\phi_{K,(j)}(\hat{\mathbf{x}})=r^{-1}\sum_{b=1}^{2N+1}\alpha_b x^b\ \ \phi_{K,(j)}(\hat{\mathbf{x}})=\phi_{-K,(j)}(\hat{\mathbf{x}}),$$

$$(\vec{\alpha}\cdot\vec{\mathbf{p}})r^{-N}\phi_{K,(j)}(\hat{\mathbf{x}})=\sum_{b=1}^{2N+1}\alpha_b p_b\ \ r^{-N}\phi_{K,(j)}(\hat{\mathbf{x}})=iKr^{-N-1}\phi_{-K,(j)}(\hat{\mathbf{x}}). \tag{24}$$

Substitution of $\Psi_{K(j)}(\mathbf{x})$ into the Dirac equation (5) leads to the following radial equation,

$$G'(r)+\frac{K}{r}G(r)=[E-V(r)-M]F(r),$$

$$-F'(r)+\frac{K}{r}F(r)=[E-V(r)+M]G(r), \tag{25}$$

where and hereafter the prime denotes the first derivative with respect to the variable $r(\rho)$.

B. The SO(2*N*) case

As we know, the reducible spinor representation of SO$(2N)$ is reduced to two inequivalent fundamental spinor representations $(+s)\equiv(0,0,\ldots,0,1)$ and $(-s)\equiv(0,0,\ldots,1,0)$. Likewise it is shown from Eq. (12) that the Casimir for both spinor representations can be obtained as $C_2[(\pm s)]=(2n^2-n)/4$. From the α_a matrices given in Eq. (23), we define the γ^μ matrices for $D=2N$:

$$\gamma^0=\alpha_{2N+1},\qquad \gamma^a=\alpha_{2N+1}\alpha_a,\qquad a\in[1,2N]. \tag{26}$$

Here the γ^0 is a diagonal matrix where half of the diagonal elements are equal to $+1$ and the remainder to -1. On considering the spinor operator S_{ab} and the operator κ are commutative with γ^0, each of them becomes a direct sum of two matrices, referring to the rows with the eigenvalues $+1$ and -1 of γ^0, respectively. The spinors $\chi_\pm(\mathbf{m})$ belong to the spinor representations $(+s)$ and $(-s)$, respectively, and satisfy

$$\gamma^0\chi_\pm(\mathbf{m})=\pm\chi_\pm(\mathbf{m}). \tag{27}$$

Thus the product of $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})$ and $\chi_\pm(\mathbf{m}')$ belongs to the direct product of two representation (l) and $(\pm s)$, which is a reducible representation:

$$(l)\times(+s)\simeq(l,0,\ldots,0,1)\oplus(l-1,0,\ldots,0,1,0),$$

$$(l)\times(-s)\simeq(l,0,\ldots,0,1,0)\oplus(l-1,0,\ldots,0,1). \tag{28}$$

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There exist two kinds of representations for the total angular momentum: the representation $(j_1)\equiv(l,0,\ldots,0,1)$ and the representation $(j_2)\equiv(l,0,\ldots,0,1,0)$. Nevertheless, their Casimirs are equal:

$$C_2[(j_1)]=C_2[(j_2)]=l(l+2N-1)+\frac{N(2N-1)}{4}. \tag{29}$$

Similar to the case of SO$(2N+1)$, there exist two different ways to obtain the wave functions belonging to the representation (j_1): the combination of $Y^{(l)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_+(\mathbf{m}')$ and that of $Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_-(\mathbf{m}')$. Because of the spherical symmetry, one only calculates the highest weight state for the representation (j_1) by the Clebsch–Gordan coefficients:

$$\phi_{K,(j_1)}(\hat{\mathbf{x}})=Y^{(l)}_{(l)}(\hat{\mathbf{x}})\chi_+[(+s)]=N_{D,l}r^{-l}(x^1+ix^2)^l\chi_+[(+s)], \tag{30a}$$

and

$$\begin{aligned}\phi_{-K,(j_1)}(\hat{\mathbf{x}})=&\sum_{\mathbf{m}} Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_-[(j_1)-\mathbf{m}]\langle(l+1),\mathbf{m},(+s),(j_1)-\mathbf{m}|(j_1),(j_1)\rangle\\ =&N_{D,l}r^{-l-1}(x^1+ix^2)^l\{x^{2N-1}+ix^{2N}\chi_-[(-s)]+(x^{2N-3}+ix^{2N-2})\\ &\times\chi_-[(0,\ldots,0,1,\overline{1},0)]+(x^{2N-5}+ix^{2N-4})\chi_-[(0,\ldots,0,1,\overline{1},0,1)]+\cdots\\ &+(x^3+ix^4)\chi_-[(1,\overline{1},0,\ldots,0,1)]+(x^1+ix^2)\chi_-[(\overline{1},0,\ldots,0,1)]\},\end{aligned} \tag{30b}$$

with

$$K=C_2[(j_1)]-C_2[(l+1)]-C_2[(+s)]+N-\tfrac{1}{2}=l+N-\tfrac{1}{2}.$$

However, for the representation $(j_2)\equiv(l,0,\ldots,0,1,0)$ we obtain

$$\phi_{-K,(j_2)}(\hat{\mathbf{x}})=Y^{(l)}_{(l)}(\hat{\mathbf{x}})\chi_-[(-s)]=N_{D,l}r^{-l}(x^1+ix^2)^l\chi_-[(-s)], \tag{31a}$$

and

$$\begin{aligned}\phi_{K,(j_2)}(\hat{\mathbf{x}})=&\sum_{\mathbf{m}} Y^{(l+1)}_{\mathbf{m}}(\hat{\mathbf{x}})\chi_+[(j_2)-\mathbf{m}]\langle(l+1),\mathbf{m},(-s),(j_2)-\mathbf{m}|(j_2),(j_2)\rangle\\ =&N_{D,l}r^{-l-1}(x^1+ix^2)^l\{x^{2N-1}-ix^{2N}\chi[(+s)]+(x^{2N-3}+ix^{2N-2})\chi_+[(0,\ldots,0,1,0,\overline{1})]\\ &+(x^{2N-5}+ix^{2N-4})\chi_+[(0,\ldots,0,1,\overline{1},1,0)]+\cdots+(x^3+ix^4)\chi_+[(1,\overline{1},0,\ldots,0,1,0)]\\ &+(x^1+ix^2)\chi_+[(\overline{1},0,\ldots,0,1,0)]\},\end{aligned} \tag{31b}$$

with

$$K=C_2[(j_2)]-C_2[(l+1)]-C_2[(+s)]+N-\tfrac{1}{2}=-(l+N-\tfrac{1}{2}).$$

From the explicit forms of α_a one obtains

$$\begin{aligned}(\vec{\alpha}\cdot\hat{\mathbf{x}})\phi_{K,(j_\omega)}(\hat{\mathbf{x}})&=r^{-1}\sum_{a=1}^{2N}\alpha_a x^a\phi_{K,(j_\omega)}(\hat{\mathbf{x}})=\phi_{-K,(j_\omega)}(\hat{\mathbf{x}}),\\ (\vec{\alpha}\cdot\vec{\mathbf{p}})r^{-N+1/2}\phi_{K,(j_\omega)}(\hat{\mathbf{x}})&=\sum_{a=1}^{2N}\alpha_a p_a\ r^{-N+1/2}\phi_{K,(j_\omega)}(\hat{\mathbf{x}})=iKr^{-N-1/2}\phi_{-K,(j_\omega)}(\hat{\mathbf{x}}),\end{aligned} \tag{32}$$

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with $\omega=1$ or 2.

The wave functions $\Psi_{K,(j_\omega)}(\mathbf{x})$ of the total angular momentum belonging to the IR (j_ω) are expressed as

$$\Psi_{|K|,(j_1)}(\mathbf{x},t)=r^{-N+1/2}e^{-iEt}\{F(r)\phi_{|K|,(j_1)}(\hat{\mathbf{x}})+iG(r)\phi_{-|K|,(j_1)}(\hat{\mathbf{x}})\},$$

$$\Psi_{-|K|,(j_2)}(\mathbf{x},t)=r^{-N+1/2}e^{-iEt}\{F(r)\phi_{-|K|,(j_2)}(\hat{\mathbf{x}})+iG(r)\phi_{|K|,(j_2)}(\hat{\mathbf{x}})\}, \tag{33a}$$

$$\kappa\Psi_{K,(j_\omega)}(\mathbf{x})=K\Psi_{K,(j_\omega)}(\mathbf{x}), \quad |K|=l+N-\tfrac{1}{2}, \quad \omega=1 \text{ or } 2,$$

with the following properties:

$$H_1(J)\Psi_{K,(j_\omega)}(\mathbf{x})=l\Psi_{K,(j_1)}(\mathbf{x}), \qquad H_{N-1}(J)\Psi_{K,(j_1)}(\mathbf{x})=0, \tag{33b}$$

$$H_N(J)\Psi_{K,(j_1)}(\mathbf{x})=\Psi_{K,(j_1)}(\mathbf{x}), \quad H_{N-1}(J)\Psi_{K,(j_2)}(\mathbf{x})=\Psi_{K,(j_2)}(\mathbf{x}), \tag{33c}$$

$$H_N(J)\Psi_{K,(j_2)}(\mathbf{x})=0, \qquad H_\mu(J)\Psi_{K,(j_\omega)}(\mathbf{x})=0, \qquad \mu\in[2,N-2]. \tag{33d}$$

Their partners can be calculated by the lowering operators F_μ.

Substitution of $\Psi_{K(j_\omega)}(\mathbf{x})$ into the Dirac equation (5) allows us to obtain the radial equations, which are in the same forms as those in $D=2N+1$ case:

$$G'(r)+\frac{K}{r}G(r)=[E-V(r)-M]F(r),$$

$$-F'(r)+\frac{K}{r}F(r)=[E-V(r)+M]G(r). \tag{34}$$

IV. THE EXACT SOLUTIONS OF THE RADIAL EQUATION

Although the wavefunctions and the eigenvalues K are different for the $D=2N+1$ case and the $D=2N$ case, the forms of the radial equations are unified

$$G'_{KE}(r)+\frac{K}{r}G_{KE}(r)=[E-V(r)-M]F_{KE}(r),$$

$$-F'_{KE}(r)+\frac{K}{r}F_{KE}(r)=[E-V(r)+M]G_{KE}(r), \tag{35}$$

$$K=\pm(2l+D-1)/2.$$

We now consider the Dirac equation with a mixed potential including a Coulomb potential and a scalar one. The Coulomb potential is derived from the exchange of massless photons between the nucleus and the lepton orbiting around it, namely,

$$V_c=-\frac{A_1}{r}. \tag{36}$$

However, the scalar potential

$$V_s=-\frac{A_2}{r} \tag{37}$$

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is added to the mass term of the Dirac equation, which can be interpreted as an effective, position-dependent mass. It is created by the exchange of the massless scalar meson. The A_1 and A_2 are the electrostatic and the scalar coupling constants, respectively.

It is found that the radial components $F_{KE}(r)$ and $G_{KE}(r)$ satisfy the following first-order differential equations

$$\begin{aligned} G'_{KE}(r)+\frac{K}{r}G_{KE}(r)&=\left(E-M+\frac{A_1+A_2}{r}\right)F_{KE}(r),\\ -F'_{KE}(r)+\frac{K}{r}F_{KE}(r)&=\left(E+M+\frac{A_1-A_2}{r}\right)G_{KE}(r). \end{aligned} \tag{38}$$

It is convenient to introduce ρ for the bound states:

$$\rho=2r\sqrt{M^2-E^2}, \qquad |E|<M. \tag{39}$$

We thus have

$$\begin{aligned} G'_{KE}(\rho)+\frac{K}{\rho}G_{KE}(\rho)&=\left(-\frac{1}{2}\sqrt{\frac{M-E}{M+E}}+\frac{A_1+A_2}{\rho}\right)F_{KE}(\rho),\\ F'_{KE}(\rho)-\frac{K}{\rho}F_{KE}(\rho)&=\left(-\frac{1}{2}\sqrt{\frac{M+E}{M-E}}-\frac{A_1-A_2}{\rho}\right)G_{KE}(\rho). \end{aligned} \tag{40}$$

Define the wave functions $\Phi_{\pm}(\rho)$ with the forms

$$\begin{aligned} G_{KE}(\rho)&=\sqrt{M-E}[\Phi_+(\rho)+\Phi_-(\rho)],\\ F_{KE}(\rho)&=\sqrt{M+E}[\Phi_+(\rho)-\Phi_-(\rho)]. \end{aligned} \tag{41}$$

Substitutions of Eq. (41) into Eq. (40) allow us to write down

$$\begin{aligned} \Phi'_+(\rho)+\Phi'_-(\rho)+\frac{K}{\rho}[\Phi_+(\rho)+\Phi_-(\rho)]&=\left[-\frac{1}{2}+\frac{A_1+A_2}{\rho}\sqrt{\frac{M+E}{M-E}}\right][\Phi_+(\rho)-\Phi_-(\rho)],\\ \Phi'_+(\rho)-\Phi'_-(\rho)-\frac{K}{\rho}[\Phi_+(\rho)-\Phi_-(\rho)]&=\left[-\frac{1}{2}-\frac{A_1-A_2}{\rho}\sqrt{\frac{M-E}{M+E}}\right][\Phi_+(\rho)+\Phi_-(\rho)]. \end{aligned} \tag{42}$$

Their addition and subtraction lead to

$$\begin{aligned} \Phi'_+(\rho)-\left(\frac{A_1E+A_2M}{\rho\sqrt{M^2-E^2}}-\frac{1}{2}\right)\Phi_+(\rho)&=-\left(\frac{K}{\rho}+\frac{A_1M+A_2E}{\rho\sqrt{M^2-E^2}}\right)\Phi_-(\rho),\\ \Phi'_-(\rho)+\left(\frac{A_1E+A_2M}{\rho\sqrt{M^2-E^2}}-\frac{1}{2}\right)\Phi_-(\rho)&=-\left(\frac{K}{\rho}-\frac{A_1M+A_2E}{\rho\sqrt{M^2-E^2}}\right)\Phi_+(\rho). \end{aligned} \tag{43}$$

Taking the following conventions,

$$\tau=\frac{A_1E+A_2M}{\sqrt{M^2-E^2}}, \qquad \tau'=\frac{A_1M+A_2E}{\sqrt{M^2-E^2}}, \tag{44}$$

we have

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$$\Phi'_+(\rho)-\left(\frac{\tau}{\rho}-\frac{1}{2}\right)\Phi_+(\rho)=-\frac{\tau'+K}{\rho}\Phi_-(\rho),$$
$$\Phi'_-(\rho)+\left(\frac{\tau}{\rho}-\frac{1}{2}\right)\Phi_-(\rho)=\frac{\tau'-K}{\rho}\Phi_+(\rho), \tag{45}$$

from which we can obtain the following important second-order differential equations:

$$\left[\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}+\left(-\frac{1}{4}+\frac{\tau\pm 1/2}{\rho}-\frac{\eta^2}{\rho^2}\right)\right]\Phi_\pm(\rho)=0,$$
$$\eta^2=K^2-A_1^2+A_2^2. \tag{46}$$

For the weak Coulomb potential, we have

$$\eta=\sqrt{K^2-A_1^2+A_2^2}>0. \tag{47}$$

It is found that Eq. (46) is a special case of the Tricomi equation,[17] which can be expressed as

$$\frac{d^2y}{dx^2}+\left(a+\frac{b}{x}\right)\frac{dy}{dx}+\left(\alpha+\frac{\alpha}{x}+\frac{\xi}{x^2}\right)y=0. \tag{48}$$

From the behaviors of the wave functions at the origin and infinity, we define

$$\Phi_\pm(\rho)=\rho^\eta e^{-\rho/2}R_\pm(\rho). \tag{49}$$

Substitution of this into (47) leads to

$$\frac{d^2}{d\rho^2}R_\pm(\rho)+\left(-1+\frac{1+2\eta}{\rho}\right)\frac{d}{d\rho}R_\pm(\rho)+\frac{\tau-\eta-\frac{1}{2}\pm\frac{1}{2}}{\rho}R_\pm(\rho)=0, \tag{50}$$

whose solutions are the confluent hypergeometric functions

$$R_+(\rho)=a_0\Phi(\eta-\tau,2\eta+1;\rho),$$
$$R_-(\rho)=b_0\Phi(1+\eta-\tau,2\eta+1;\rho), \tag{51}$$

which imply that $G_{KE}(\rho)$ and $F_{KE}(\rho)$ can be directly expressed by the combinations of the confluent hypergeometric functions.

We now study the relation between the coefficients a_0 and b_0. Before proceeding to do so, it is necessary to review the following recursive relations between the confluent hypergeometric functions[17]

$$\gamma\frac{d}{dz}\Phi(\alpha,\gamma;z)=\alpha\,\Phi(\alpha+1,\gamma+1;z),$$
$$z\Phi(\alpha+1,\gamma+1;z)=\gamma\Phi(\alpha+1,\gamma;z)-\gamma\Phi(\alpha,\gamma;z),$$
$$\alpha\Phi(\alpha+1,\gamma+1;z)=(\alpha-\gamma)\Phi(\alpha,\gamma+1;z)+\gamma\Phi(\alpha,\gamma;z), \tag{52}$$
$$\alpha\Phi(\alpha+1,\gamma;z)=(z+2\alpha-\gamma)\Phi(\alpha,\gamma;z)+(\gamma-\alpha)\Phi(\alpha-1,\gamma;z).$$

It is shown from Eqs. (46), (51) and (52) that

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$$\left(\frac{\eta-\tau}{\rho}a_0+\frac{\tau'+K}{\rho}b_0\right)\Phi(1+\eta-\tau,2\eta+1;\rho)=0. \tag{53}$$

Since both a_0 and b_0 cannot be vanishing, we obtain

$$b_0=\frac{\tau-\eta}{\tau'+K}a_0. \tag{54}$$

From Eq. (42) we thus have

$$\begin{aligned} G_{KE}(\rho)=&N_{KE}\sqrt{M-E}\rho^{\eta}e^{-\rho/2}\\ &\times[(\tau'+K)\Phi(\eta-\tau,2\eta+1;\rho)+(\tau-\eta)\Phi(1+\eta-\tau,2\eta+1;\rho)],\\ F_{KE}(\rho)=&N_{KE}\sqrt{M+E}\rho^{\eta}e^{-\rho/2}\\ &\times[(\tau'+K)\Phi_1(\eta-\tau,2\eta+1;\rho)-(\tau-\eta)\Phi(1+\eta-\tau,2\eta+1;\rho)], \end{aligned} \tag{55}$$

where the normalization factor $N_{KE}=a_0(\tau'+K)^{-1}(2\sqrt{M^2-E^2})^{-1/2}$ can be determined later.

We now study the eigenvalues of this quantum system. The quantum condition is obtained from the finiteness of the solutions at infinity:

$$\tau-\eta=n'=0,1,2,\ldots, \tag{56}$$

when $n'=0$, $\eta=\tau$, and

$$K^2=\tau^2+A_1^2-A_2^2=(\tau')^2.$$

Therefore K has to be positive in order to avoid the trivial solution.

Introducing the principal quantum number

$$n=|K|-(D-3)/2+n'=|K|-(D-3)/2+\tau-\eta=l+1+n'=1,\ 2,\ldots, \tag{57}$$

we have

$$\frac{EA_1+MA_2}{\sqrt{M^2-E^2}}=n-|K|+\frac{D-3}{2}+\eta=n'+\eta\equiv\kappa. \tag{58}$$

The energy E can be solved from Eq. (58):

$$E(n,K)=M\left\{-\frac{A_1A_2}{A_1^2+\kappa^2}\pm\left[\left(\frac{A_1A_2}{A_1^2+\kappa^2}\right)^2-\frac{A_2^2-\kappa^2}{A_1^2+\kappa^2}\right]^{1/2}\right\}. \tag{59}$$

We now consider a few special cases. First, if $A_1=0$, then $\eta=\sqrt{K^2+A_2^2}$, and

$$E(n,K)=\pm M\left(1-\frac{A_2^2}{\kappa^2}\right)^{1/2}. \tag{60}$$

It implies that there are two branches of solutions symmetric for the positive and negative energies. For a large D, we have

$$E(n,D)\simeq\pm M[1-2A_2^2D^{-2}+4A_2^2(2n-3)D^{-3}-\cdots], \tag{61}$$

which implies that the energy is independent of l for a large D. For a small A_2, we have

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$$E(n,l,D)\simeq \pm M\left\{1-\frac{A_2^2}{2[n+(D-3)/2]^2}+\frac{A_2^4}{2[n+(D-3)/2]^4}\left(\frac{2n+D-3}{2l+D-1}-\frac{1}{4}\right)\right\}, \tag{62}$$

where the first term on the right-hand side is the rest energy M ($c^2=1$ in our conventions), the second one is from the solutions of the Schrödinger equation, and the third one is the fine structure energy, which removes the degeneracy between the states with the same n.

Second, if $A_2=0$, then $\eta=\sqrt{K^2-A_1^2}$ and from Eq. (58) E has the same sign as A_1 when $K^2>A_1^2$. For the attractive Coulomb potential ($A_1>0$) we have the positive energy E_{nK}

$$E_{nK}=M\left(1+\frac{A_1^2}{\kappa^2}\right)^{-1/2}. \tag{63}$$

It coincides with the conclusion from the Sturm–Liouville theorem for a weak attractive potential.[18] For a large D similarly we have the same result as Eq. (61).

For a small A_1, we have

$$E(n,l,D)\simeq M\frac{A_1}{|A_1|}\left\{1-\frac{A_1^2}{2[n+(D-3)/2]^2}-\frac{A_1^4}{2[n+(D-3)/2]^4}\left(\frac{2n+D-3}{2l+D-1}-\frac{3}{4}\right)\right\}. \tag{64}$$

Similarly, the physical meanings of three terms are similar to those of Eq. (62) except for the different expansion coefficients.

We are now briefly considering the special case $D=1$ in this case. It is found that there is absence of the bound states since η becomes imaginary regardless of the value of A_1. This can be easily checked from the fact that the eigenvalues and eigenfunctions do not exist at all.

Third, if $A_1=A_2$, we have $\eta=|K|$ and

$$E(n)=M\left[-\frac{A_1^2}{A_1^2+(n+(D-3)/2)^2}\pm\frac{(n+(D-3)/2)^2}{A_1^2+(n+(D-3)/2)^2}\right]. \tag{65}$$

If we choose the negative sign in the result, we have $E=-M$, which is a singular solution of Eq. (58). For the positive sign, we have

$$E(n)=M\left[1-\frac{2A_1^2}{A_1^2+(n+(D-3)/2)^2}\right]. \tag{66}$$

We now determine the normalization factor N_{KE} from the normalization condition

$$\int \Psi_{KE}^{\dagger}\Psi_{KE}dV=1. \tag{67}$$

Noticing $n'=\tau-\eta$ is a non-negative integer, we can express the confluent hypergeometric function by the associated Laguerre polynomial[19]

$$L_n^{\alpha}(\rho)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\,{}_1F_1(-n,\alpha+1;\rho), \tag{68}$$

$$\int_0^{\infty}\rho^{\alpha}e^{-\rho}L_n^{\alpha}(\rho)L_m^{\alpha}(\rho)d\rho=\frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}. \tag{69}$$

Through a direct calculation we obtain

$$N_{KE}=\frac{(M^2-E^2)^{1/4}}{\Gamma(2\eta+1)}\left[\frac{\Gamma(\tau+\eta+1)}{2M\tau'(K+\tau')(\tau-\eta)!}\right]^{1/2}. \tag{70}$$

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V. CONCLUDING REMARKS

In this work we have studied the (D+1)-dimensional Dirac equation with a Coulomb plus scalar potential with the interest of higher-dimensional field theory. The eigenfunctions can be analytically obtained by studying the second-order differential equations obtained from the first-order coupled ones. The eigenvalues as well as their special cases are studied. Before ending this article, we give two remarks here. First, in comparison with the 3D case, the angular momentum quantum number K in $D+1$ dimensions plays the role of the good quantum number κ in three dimensions (more strictly, $|K|+\frac{1}{2}\leftrightarrow|\kappa|$). Second, for the special case $D=1$, it is found from Eq. (35) that $K=0$. Therefore, it is shown from Eq. (47) that η becomes imaginary if $|A_2|<|A_1|$, which means that there is absence of the bound states in this case. On the contrary, there exist the bound states if $|A_2|>|A_1|$.

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